Nonlinear Robust Control Theory and Applications

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Integrated modeling, identification, analysis, and design.

The use of mathematical models to design large systems involves various instances and possibly iterations of modeling, identification, system design and controller synthesis, and various forms of analysis and simulation. Critical to this endeavor is the mathematical machinery to systematically *address* as many stages of this process as possible. Automating this process is the focus of the development of theory for virtual engineering (VE).

The various VE activities are currently performed using a variety of mathematical machinery, but we have recently developed an extremely promising unified framework for these various aspects of system design. In addition to providing an "interface" between system ID and control, it has the potential to overcome the major deficiencies in conventional system identification: the difficulty of using a priori information and rich uncertainty descriptions and still obtain global solutions for parameter estimates. This framework also allows advances in different directions to combine readily. For example, it is clear how to combine progress in linear system ID with progress in nonlinear robustness analysis to produce nonlinear system ID methods, a major need in VE. Our framework also clarifies many of the computational issues and what basic algorithms must be developed.

In addition to this unification of the system ID and control process, many new tools are developed to address a richer class of robustness analysis questions. Traditional robustness questions addressed in the structured singular value, μ, framework have addressed worst case formulations. The same tools can be used to perform probabilistic robustness analysis but in general the computation is intractable and we have developed new tools which may be useful in reducing the computational growth. These new tools are for the worst-case analysis of more exotic descriptions of uncertain models. We have extended the μ-framework for analyzing certain robustness properties of various classes of nonlinear systems. These tools

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This research has developed a suite of fundamental tools for automating ithe system design, system identification,, controller synthesis, and various aspects of analysis and simulation.

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are particularly useful for the analysis of real engineering systems which operate in various regimes or along particular trajectories. The μ approach to robustness analysis blends nicely with $\mathcal{H}_\infty$ specifications of performance requirement, but of more engineering interest is the $\mathcal{H}_2$ specifications of performance. We have made great strides in the analysis of the robust $\mathcal{H}_2$ problem both from a theoretical and practical viewpoint.

For complex systems it is not tractable to use the highest fidelity models to investigate system level performance, there is too much computation associated with detailed behavior when the dynamics of interest are macroscopic. Further in the process of control design, complex models lead to unnecessarily complex controllers and are a waste of the computational power available and from experience these complex controllers tend to be less robust in practice. We have developed a systematic method for the reduction of uncertain models to distill out the important macroscopic behavior of the system and to cover the approximations of the new model by an uncertainty description.

In the process of optimal control design, it is necessary to specify a performance requirement. We have generalized the $\mathcal{H}_\infty$ synthesis solution to address a broader class of performance specifications. Including robust $\mathcal{H}_2$ and square $\mathcal{H}_\infty$. W e have extended these tools for control design for LTI systems to linear time-periodic systems. We have also investigated various nonlinear control design methodologies to understand why they work so well in practice given enough tweaking of the problem formulation and how some methodologies can be blended to produce a continuum of control strategies as one method is traded off with another.

Robustness analysis computation.

At the heart of our unified framework is the robustness analysis of uncertain systems. In the last several years, a much clearer picture of the computational implications of various uncertainty assumptions has emerged. While a detailed description is not possible here, one important observation is that robustness analysis with real parametric uncertainty is NP hard, generally viewed as implying worst-case intractability. Except for special cases, the more general methods of identification and implicit analysis are also NP hard. Conventional numerical analysis notions of guaranteed algorithm convergence are irrelevant in these problems, because global convergence is computationally prohibitive and local convergence is of little value. Thus the only reasonable strategy is to aim for algorithms which exhibit experimentally good performance on problems of engineering interest, and here our success with extensive numerical experience in robustness analysis is very encouraging. A unified framework allows for experience gained in algorithms for one particular problem to be transferred to the more general class.

Thus, to obtain acceptable computation, we do not attempt to solve the various hard problems exactly but rather to obtain good bounds, and aim for acceptable growth rates on problems of engineering interest, rather than for all problems that are mathematically possible. Upper bounds are usually convex feasibility problems, which have tractable computation. We have demonstrated that branch-and-bound can be successfully used to overcome the intractability of mixed μ, and developed power algorithms for the lower bounds which

are much faster and produce better bounds than conventional local optimization [34, 35, 36].

Several new approaches to computing an improved μ lower bound have been presented in [24]. These algorithms have been combined to yield a substantially improved power algorithm. The nature of the mixed μ problem is such that the only meaningful way to evaluate an algorithm is by testing it on a large number of representative problems. We compared proposed algorithms, each run on the same type of problems, and showed how the performance of the best algorithm depends on problem size.

For branch and bound to be effective, it is essential that the bounds get reasonably close without extensive subdivision, because just splitting each real parameter once would still yield 2^p subdomains. Thus pruning must eliminate most branches, essentially preventing the tree from getting too broad. The evidence so far, gathered on many thousands of examples chosen to be representative suggests that branch and bound can be used to get the worst-case bounds ratio as good as the average bound ratio with modest additional computational cost. For this scheme to work, it is essential that the bounds have a good average ratio. As we extend explicit uncertainty modeling to new domains, this will remain a critical research problem.

When the branch and bound method is applied to the worst-case computation, only axially aligned cuts were performed. Although more intelligent branching schemes can be explored, the computational experience is it's not as critical as improving the original quality of the bounds. The need for performing non-axial cuts arose in the probabilistic robustness analysis, which aims at providing hard bounds on the probability distribution of a system's performance assuming the distribution of the uncertainty is given, which is a complement to the "soft" bounds provided by traditional statistical methods, such as Monte Carlo simulation and importance sampling. Our another goal is using branch and bound to tremendously increase the effective number of trials in the Monte-Carlo simulation to achieve high confidence levels when assessing rare events.

Probabilistic robustness analysis is computationally more challenging than the worst case analysis. Our experience is simple application of the branch and bound schemes with axial cuts didn't break the intractability of the problems. ([37],[18]) And in this case, better quality of the bounds doesn't help much. The intuition is that exponential growth in the number of branches is inevitable if the branching is not appropriately aligned with the boundary of singularity. For the rank-one problems the boundary is linear in the uncertain parameters, therefore a linear cut along the boundary will be much more effective to exclude benign regions than the cuts aligned with the axes. Numerical experiments also showed that general random matrices behave like rank-one matrices near the worst-case. This observation motivated us to investigate the computation of μ with richer classes of uncertainty descriptions.

Spherical μ deals with the uncertainty set described by a spherical constraint rather than the standard ∞-norm constraint. In [19], an upper bound to spherical μ with only nonrepeated complex scalars can be computed by solving an LMI analogous to the one in the standard μ case. The main difference between the standard and spherical upper bounds is that they involve different quadratic forms in the signal space description. This results in a slightly different optimization problem associated with the upper bound. A closed form expression for the solution optimization problem can be derived [27]. The upper

bound optimization problem is further generalized to uncertainty with ellipsoidal constraints, repeated scalars, real parameters, and full blocks.

Linear constraints can be viewed as the extreme of the ellipsoidal constraints when the eccentricity goes to infinity. An upper bound to μ with linear cuts is computed using a combination of the ellipsoidal constraint and the standard ∞-norm constraint. Another way to handle linear constraints is to use implicit method. An implicit system with higher dimension can be constructed to incorporate the linear term into the constraints on the parameter set, then an upper bound can be obtained by solving the LMI for the corresponding implicit μ upper bound. Numerical experiments compared these two methods with another one called the parallelogram method and the result is that the implicit method gives best bounds for most problems, [38]. Future research involves applying this method to the probabilistic framework to access the probability of rare events. For higher dimensional problems multiple linear cuts will be necessary, therefore the computational issues involved in implementing multiple linear constraints by the implicit method need to be studied.

A single μ robustness performance analysis provides a bound β on the uncertainty under which stability as well as $\mathcal{H}_\infty$ performance level $\frac{1}{\beta}$ are guaranteed. While this approach does provide a stability and performance margin, a good estimate of the actual uncertainty bound may be available. In that case, assuming that the uncertainty bound has been normalized, a question of interest is whether the system is stable, whenever the uncertainty has size less than 1, and if that is the case, what is the worst case performance for this same uncertainty size.

In [13] we considered a class of uncertain systems subject to a norm bounded structured LTI perturbations. We showed that the worst case $\mathcal{H}_\infty$ gain of a system can be written exactly in terms of the skewed structured singular value. Although, like μ, the skewed structured singular value can not be computed exactly, we discussed efficient algorithm to compute corresponding upper and lower bounds. The results presented show that the enhanced algorithm developed recently for the structured singular value, can be extended to the problem of computing worst case gains under fixed size uncertainty, without significant loss of performance or accuracy.

In many cases, considering a slightly different version of previously described system allows us to set additional robust performance questions in the μ framework. For example, the problem of computing the worst case $\mathcal{H}_2$ norm of an uncertain system has always been considered an important one, since many useful performance requirements are captured by it. Many recent publications have presented different approaches to solving this problem. However, in all cases, the results developed provide only upper bounds on the given norm when the uncertainty is linear time invariant.

In [31] we presented how the worst case $\mathcal{H}_2$ norm of an uncertain system subject to norm bounded structured LTI perturbations can be written exactly in terms of the complex skewed structured singular value. Even though computation of the lower and upper bound for the worst case $\mathcal{H}_2$ performance implies numerical integration over frequency, which is the same as in a frequency by frequency evaluation of the structured singular value (the current practice for robustness analysis in industry), this new approach can be effortlessly integrated into the current robustness analysis tools.

Some extensions of the structured singular value (μ) theory to more general uncertainty descriptions have been developed [17]. In these descriptions, the allowable uncertainty set Δ is characterized through generalized integral quadratic constraints (IQCs, [6, 22]).

Convex upper bounds are obtained, analogous to the standard LMI(linear matrix inequalities) μ upper bounds. In some cases, these bounds can be computed very efficiently (without solving the LMIs), since the optimal value can be shown to be equal to the spectral radius of an associated linear operator [27].

These LMIs belong to a particular class, called cone-preserving linear matrix inequalities, for which the corresponding theory has been developed. By using a generalized version of the classical Perron–Frobenius theorem, the optimal value is shown to be equal to the spectral radius of an associated linear operator. This allows for a much more efficient computation of the optimal solution, using for instance power iteration-type algorithms. We are currently investigating to what extent these convenient numerical properties can be exploited in the important cases where only part of the LMI has the cone-invariance property.

In parallel with the theoretical work described above, a preliminary study on the suitability of the current uncertainty analysis tools to nonlinear problems has been started. Designed as a exploratory analysis from a computational viewpoint, the initial stages are primarily aimed to determine the kind of system properties that can be determined from the knowledge of uncertain models. In other words, what properties are robust under uncertainty, and how can they can be computed.

Analysis of implicit systems.

Implicit models play a key role in this unified framework. We have extended robustness analysis techniques to systems described in implicit form, developing new tools for analysis of systems under a combination of time-invariant/time-varying perturbations and exact conditions for robust H_2 performance analysis. There is strong engineering motivation for this extension, particularly for VE. In fact, the standard control theory I/O formulation is only adequate for systems which are deliberately built to match the "signal flow" conception. It appears awkward when modeling physical systems from first principles, where physical laws such as mass, momentum, or energy balances or physical laws such as Newton's second law, Ohm's law, and so on are more naturally thought of as relations between variables than as I/O maps. This is entirely compatible with the behavioral framework of Willems, and much of our work recently has focused on integrating this framework with robust control.

The important uncertainty modeling machinery from robust control can not only be generalized appropriately to implicit systems, but also greatly extended to treat entirely new problems. The implicit form analysis allows for over-constrained problems, such as those involving an uncertain system and a finite number of integral quadratic constraints (IQCs), which may be used to obtain richer signal characterizations. It also provides a framework for the formulation of model validation/system identification questions. For systems with structured uncertainty involving a combination of linear time-invariant and linear time-varying perturbations, an exact test for analysis was obtained, based on a finite augmentation of the original problem. Conditions based on scaled small-gain are also available for this case,

and the class of perturbations for which these conditions become necessary has been characterized. A necessary and sufficient condition was obtained for worst-case H_2-performance analysis under structured uncertainty. This test is a convex feasibility condition across frequency, of the same nature and computational complexity as the corresponding conditions for H_∞ performance. The proof is based on a deterministic characterization of white noise signals, and the necessity proofs involve an extended "S-procedure losslessness" result on quadratic functions on L_2.

Component Synthesis.

Just as it is natural to adopt implicit models in the analysis of complex systems, large gains may be obtained by developing synthesis methods which may be applied to systems without signal flow graphs. A system is viewed as a family of allowable trajectories, and the design objective becomes the synthesis of a component which when interconnected with the given system, further restricts the allowable trajectories such that pre-specified performance objectives are satisfied. Thus the component to be designed is itself a system, with no pre-specified signal flow graphs. In [4], the H_∞ framework was extended to encompass these more general types of optimization problems in the absence of uncertainty.

Extensions of this approach to general systems described in implicit form will have a substantial impact on VET, since parts of the design process may be automated. The various technical hurdles present in the analysis of implicit systems are inherited by the synthesis problem. In fact, the synthesis problem is typically much harder, and various techniques will have to be developed in order to construct tractable algorithms.

Model validation/ID.

The extension of our analysis algorithms to the implicit formulation is currently being pursued, as are issues arising from the incorporation of data, both of which are necessary for solving ID/model validation problems. Although results with a preliminary algorithm are very encouraging, development is needed to further improve the convergence properties as was done for the standard case. An LMI upper-bound has recently been coded in matlab using the LMI-Lab toolbox. Application of these bounds to computation for experimental control problems at Caltech has recently begun. This work should help lead to an identification methodology appropriate for robust control [25, 23]. Additional research at Caltech has focused on developing an approach for time-domain model validation of an uncertain, noisy continuous-time model with a discrete and finite data record; which is the most directly relevant type of model validation for control as well as VE. Validation conditions in the presence of both LTI and LTV uncertainty have been derived which are convex and can be evaluated by LMI methods, have coded optimization software to implement it, and have successfully applied the method to several laboratory experiments [8, 9, 10, 29].

Abstract data type.

Linear Fractional Transformations (LFTs) have proven to be useful in generalizing standard linear state space systems to include uncertainty (see [2] and the references therein). The engineering motivation for this work is based on the desire to have a standard framework for modeling uncertain systems that gives the benefits of state space descriptions, both for computational and theoretical convenience. In building hierarchical system models from components, the goal is to have convenient basic building blocks, with models of resistors, capacitors, masses, springs, ducts, valves, etc., and an abstract data type for their representation which naturally and explicitly includes an uncertainty representation.

We want to make the weakest possible assumptions about uncertainty in our data type, which can be strengthened in the analysis stage to allow a wide variety of uncertainty assumptions. Thus the framework we have adopted is the LFT on noncommuting indeterminants or operators. Given this framework, the major results in linear systems theory have be generalized over the last several years. Stability and L_2 gain can be characterized in terms of generalized Lyapunov inequalities, called linear matrix inequalities (LMIs), exactly as in the standard case. For realization theory, minimality can be characterized in terms of controllability and observability, all minimal realizations are related by similarity, and for nonminimal realizations there is a direct generalization of the Kalman decomposition. Balanced truncation model reduction has guaranteed error bounds, exactly as in the 1D case. For synthesis, output feedback stabilization can be reduced via a separation argument to full information and full control problems, which can be solved using LMIs. Finally, H_∞ optimal control can be generalized to this setting, again with a separation structure and 2 LMIs with a convex coupling condition. All of these results are reviewed in [2].

The natural construction of the hierarchy for a system of models is inherently tree structured. The tree structure defines a partial ordering of the models. Taking the natural reductionist approach, one has high level models. If these models are inadequate then the model must be refined into its components. This results in a reticulated model for a component, where the component model has two modes. The first mode is as a leaf of the hierarchy tree. For this mode, the model of the component contains parameters, dynamics, and uncertainty and is represented by an LFT system. The second mode is the reticulation, where the model of the component is a node of the hierarchy tree and describes the interconnection of subcomponents. Again, the model is represented by an LFT system but it no longer contains parameters, dynamics, and uncertainty. This information is contained in the subcomponents for which it defines the connection.

This sort of model construction may be useful for particular problems but it has it's flaws. Although this construction may seem completely general, but an implicit assumption is made in the construction that limits the possible hierarchies of models. The restrictive assumption is that the more accurate (reticulated) models for a component connect with the outside world using the same interconnection variables. Although this seems like a plausible restriction, it is problematic in dealing with continuum phenomenon and boundary conditions. As an example, a high level model for an electronic circuit would be represented by circuit equations, but at the lowest level it is necessary to solve Maxwell's equations to

more accurately describe the dynamics of the system. This would lead to more complex interaction between other circuits to which it is connected or in near proximity.

The implication is that there are limits to the theoretical statements which can be made about this sort of structure and that it will be necessary to employ engineering judgment and approximation to implement this sort of hierarchical models.

Model reduction.

Model based control methods are commonly used in the design of large, complex systems. For the purposes of feedback control highly accurate models are desired. However, such accuracy often requires that complicated high-order models be used, which in turn lead to more difficult control design problems from both an engineering and a computational perspective. A fundamental limitation in achieving desired system performance via any control design process is the inherent uncertainty in modeling the dynamics of the system under consideration. In [1], model reduction methods and realization theory for uncertain systems are developed, which are aimed at facilitating subsequent control design and analysis. The uncertain systems are represented by a Linear Fractional Transformation (LFT) on a block diagonal uncertainty structure. A complete generalization of balanced realizations, balanced gramians and balanced truncation model reduction with guaranteed error bounds is given, which is based on computing solutions to a pair of Linear Matrix Inequalities (LMIs). A necessary and sufficient condition for exact reducibility of uncertain systems, the converse of minimality, is also derived. This condition further generalizes the role of controllability and observability gramians, and is expressed in terms of singular solutions to the same LMIs. The reduction methods provide a systematic means for both uncertainty simplification and state order reduction in the case of uncertain systems, but also may be interpreted as state order reduction for multi-dimensional systems.

Generalized l_2 Synthesis.

In the standard $\mathcal{H}_\infty$ paradigm, the allowable disturbance class consists of arbitrary unit l_2 norm signals, while the design objective is to ensure that all output errors have l_2 norm less than one. The $\mathcal{H}_\infty$ design which is robust to plant uncertainty and insensitive to plant parameters can be performed in a systematic and rigorous fashion. As the physical interpretation of $\mathcal{H}_\infty$ optimization is the minimization of a system's power to power gain, it is implicitly assumed in the design process that the worst case disturbance is allowed to be an arbitrary power signal, such as a sinusoid. This is clearly not a proper choice for many types of physical disturbances, such as sensor or thermal noise, wind gusts, and impulsive forces. In [6], the development of the extension of the $\mathcal{H}_\infty$ optimization for LTI systems to allow for more general closed-loop design objectives is presented. From a practical point of view, various open problems can be solved using the generalized framework. From a purely theoretical standpoint, these results extend the boundary for which optimization in the l_2 framework results in computationally tractable solutions.

Under the new framework, the allowable disturbance class and the design objectives are generalized to encompass a wider class of optimization problems. The underlying signal space is still taken to be l_2; as opposed to the standard $\mathcal{H}_\infty$ synthesis, however, the allowable disturbance set and performance objective are general functions of the various inner products of the input and output variables. The constraints used to define the allowable disturbance set are very closely related to IQC's. An analysis condition is derived, which takes the form of an operator inequality. Under this condition, a method for constructing controllers that meet the performance objectives is presented, which takes the form of an affine matrix inequality (AMI). Various problems are solved using the generalized l_2 synthesis formulation: synthesis for independently norm bounded disturbances, robust stability with "element by element" bounded structural uncertainty, and certain classes of robust performance problems. In addition, recent results on the design of gain scheduled controllers are extended to the above cases.

The generalized l_2 synthesis is also extended to allow deterministic noise disturbances. One common problem of $\mathcal{H}_\infty$ controllers is that they tend to be sluggish and overly conservative. The reason is that $\mathcal{H}_\infty$ design minimizes the energy to energy gain; in many applications, modeling the disturbances as arbitrary signals is a poor choice. In contrast, the $\mathcal{H}_2$ design which minimizes the power output when the disturbances are assumed to be white noise and impulses often leads to a less conservative performance. The potential problem with $\mathcal{H}_2$ designs is that they lack robustness properties. A desirable control design strategy would then be one which has the I/O gain interpretation of the $\mathcal{H}_2$ norm, but can readily accommodate $\mathcal{H}_\infty$ bounds on the uncertainty. With white noise signals being captured in a deterministic setting [26], the generalized l_2 formulation is extended to solve the so-called mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ problem and robust $\mathcal{H}_2$ synthesis when the disturbances class consists of a mix of l_2 bounded signals and deterministic noise signals. These linear matrix inequality based solutions are non-conservative and computationally tractable. The solution of the mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ problem can, in turn, be used as one of the two iterative steps for robust $\mathcal{H}_2$ synthesis, truly putting robust $\mathcal{H}_2$ synthesis on the same par as robust $\mathcal{H}_\infty$ synthesis in terms of computational complexity.

Analysis and synthesis of time-varying systems.

Dr. Lall and outside collaborators have developed a new framework and techniques for uncertainty analysis and control of nonlinear systems along *trajectories.* This approach has focused on using linear time-varying (LTV) systems as representations of nonlinear systems along prespecified trajectories. The major attraction of this approach is both analytical and computational, because LTV systems are substantially simpler than general nonlinear systems, and the resulting approach is extremely suitable for simulation-based design and analysis.

For many nonlinear systems, it is desired to apply control to maintain system performance along a specific trajectory, or set of trajectories, in state space. Along such trajectories the behavior of a nonlinear system can be characterized as a linear time-varying system. LTV systems also arise in a more general nonlinear setting when analyzing system behavior and the

effects of uncertainty along a trajectory; this is a particularly useful approach because it can combine the information generated by a high-fidelity nonlinear simulation with powerful and computationally tractable analysis methods. Dr. Lall and collaborators [7] have developed a new mathematical framework for the analysis of LTV systems. Using this framework, such systems can essentially be treated as if they were time-invariant, with common notions such as that of frequency being well-defined. This provides a direct method of derivation of solution for analysis and synthesis problems for LTV systems. In particular, in [7], a complete solution to the H-infinity problem is given for LTV systems.

Further, previous methods which were restricted to linear time-invariant systems have been generalized to LTV systems using this framework. Dr. Lall et al [20] have derived guaranteed error bounds for the model reduction of uncertain LTV systems. This provides the first systematic way to reduce nonlinear uncertain systems along trajectories, and may find wider application in nonlinear model reduction.

Another area of research which has received attention recently is a generalized class of hybrid system known as jump systems. For these systems, Dr Lall et al [21] have developed new techniques for analysis and synthesis, making use of the LTV framework of [7]. In particular this paper solves the outstanding problem of L_2 induced norm minimization for asynchronous multi-rate sampled-data systems. Previously, only approximate solutions were known to this problem and this paper provides the first solution to asynchronous problems of this type. A further consequence is that the proofs specialize to give the simplest synthesis proof known for standard sampled-data systems.

Robust control and nonlinear extensions.

So far, the most successful applications of robust control techniques have occurred in problem domains (flexible structures, flight control, distillation) where there may be substantial uncertainty in the available models, and the degrees of freedom and the dimension of the input, output, and state may be high, but the basic structure of the system is understood, the uncertainty can be quantified. Nonlinearities are bounded and treated as perturbations on a nominal linear model, or handled by gain-scheduling linear point designs.

The state of the art in industry, as discussed above, still consists in obtaining lower bounds to the performance indices through extensive simulation or local optimization techniques. However these methods require large amounts of computation; standard optimization techniques fail even for small problems, and a search over parameter space exhibits exponential growth with the number of parameters. The methods actually used in industry share two main characteristics: performance specifications are made over a finite time horizon, and the interface between the analysis method and the system is a simulation.

In recent work at Caltech, we have begun to extend the robustness analysis techniques of linear systems, and in particular the associated computational methods, to nonlinear systems. Given the diversity of nonlinear behavior, it is clear that this cannot be done in complete generality and still maintain the efficiency and usability of the methods. We have focused on analysis methods for a specific nonlinear robust performance problem: tracking a trajectory in the presence of noise and uncertainty. Many nonlinear analysis problems of

engineering interest can be reduced to such a problem. A common example is an airplane performing an automatic change of altitude and heading. The pilot enters the new heading and altitude and the flight computer determines nominal commands to perform it. A second control loop maintains the airplane around the planned trajectory. The designer has in mind an appropriate path to be completed in a finite predetermined time, and designs the control system accordingly. Since the real system is not exactly the one used for the design, and since it is also subject to noise, the system will not follow the intended trajectory. The question of interest becomes: will the real trajectory, under the worst conditions possible, remain close enough to the nominal one in an appropriate norm?

In [30] we presented a power algorithm to compute a lower bound on the performance index associated with the robust trajectory tracking problem (i.e., the distance from the actual to the nominal trajectory). This algorithm is similar in nature to the one developed for the structured singular value, and has similar behavior. Since, as was the case for linear systems, the algorithm is not guaranteed to converge in general, its analysis is done empirically. We test this algorithm by applying it to simulations of real systems and have carried out several different performance tests on two different platforms: the Caltech ducted fan experiment and a simplified model of an F-16 jet fighter. The results of these tests are reported in [30]. These results indicate that without significant additional computation, and avoiding computationally expensive parameter searches, a lower bound on the given performance index can be computed that gives more information on the worst case behavior of the system than the standard Monte Carlo procedures.

We have also begun to develop computable upper bounds for the trajectory generation problem. One method is to use rational approximations to nonlinear systems [32]. At present, these results are still far from being practical, but they are a starting point in developing computational machinery for performing robust modeling of nonlinear systems. This effort will be a major focus of our VET research, and will leverage ongoing research in nonlinear controls. What will be particularly important for VE is to develop methods that deal with uncertainty and nonlinearity in hierarchical, multiresolution models of heterogeneous systems.

In [12] we proposed a numerical algorithm for analysis of disturbance rejection for nonlinear systems. The algorithm seeks solutions to Euler-Lagrange equations and is similar to power algorithms for μ analysis lower bound. Indeed, for linear systems we demonstrated how the newly developed algorithm reduces to a well studied algorithm for the lower bound of μ, and the algorithm is guaranteed to converge to the global optimum.

We evaluated the algorithm accuracy via converse Hamilton-Jacobi-Bellman (HJB) method which can generate the worst case disturbance for the optimal controller, but there may be even worse disturbances for non optimal controllers. The worst case disturbances obtained by our proposed power algorithm are very close to the worst case disturbances a priori given by other methods. For the general case of a system with a non-optimal controller this algorithm can provide us with knowledge of the worst case disturbance.

Robustness for nonlinear systems was proved to be equivalent to the existence of solution to Hamilton-Jacobi equations or nonlinear matrix inequalities. However, computational methods to establish the existence of these solutions have not been developed to a level

comparable to their linear counterpart (i.e., existence of solutions of Riccati equations and linear matrix inequalities), and are theoretically intractable, even for the cases which are easy for linear systems. In practice, designers have to resort to the approximation techniques. The frozen Riccati method (FRE) [3] that previously rested on shaky theoretical grounds was one of them. This technique has recently gained popularity among some of the control engineers due to its computational simplicity when compared to other techniques. Frozen Riccati techniques are so named because they rely on a state-dependent linear representation of input-affine nonlinear systems, which are used to reduce the Hamilton-Jacobi equation to a state dependent Riccati equation which can be solved on-line in a pointwise (or "frozen") fashion. Surprisingly, even though no substantial results existed to guarantee even mere stability, in practice it seemed to perform unjustifiably well. Our investigation turned up new results [15, 14], leading to at least some explanation of the promising results obtained in practice. While the state dependent linear representation of the nonlinear system is not unique, it was proven that under mild assumptions, there always exists a representation that will recover the optimal controller. Unfortunately, it was also shown that finding the proper representation is just as difficult as solving the Hamilton-Jacobi partial differential equation. Even though we have examples demonstrating that frozen Riccati techniques can produce instability, its unexpectedly good performance on many systems remains somewhat of a mystery, and deserves further examination.

While the FRE is a primitive method that uses linear techniques to tackle nonlinear systems, the problems associated with it also arise in other more sophisticated techniques, such as the linear parameter varying (LPV) scheme ([33]). As was revealed in our research, a close connection exists between the FRE and LPV methods. Both can be understood as the search for a Lyapunov or storage function without gridding the entire state space, however the LPV design guarantees stability at a price of heavier computation. In addition, the LPV method reduces to the FRE when the rate variation of the parameters is set zero. Recognizing that a critical element in both schemes is the choice of linear state dependent representation of the nonlinear system, techniques are currently being developed that incorporate this design freedom into the control selection process in an optimal fashion.

Neither of the above two techniques, FRE or LPV, exploits on-line computation. This is in contrast to a technique known as Model Predictive Control (MPC) which relies completely on on-line optimizations. Unfortunately, MPC takes an extreme point of view, and fails to benefit from information obtained off-line from LPV or FRE analysis. Furthermore, it is generally lacking any guarantee of stability, without imposing stringent or unnatural constraints. Our research revealed that off-line analysis from other techniques could aid on-line computation and solve many of these difficulties [28]. In fact, when viewed in the correct framework, most control laws obtained from off-line analysis can be viewed as limiting cases of on-line MPC schemes. Or said differently, off-line approaches, such as LPV, naturally admit an on-line extension to an MPC scheme. A unifying concept underlying these results is that of a Control Lyapunov Function (CLF), which is the generalization of a Lyapunov function to a control system [11]. All stabilizing schemes produce a CLF, and the information contained in this function essentially summarizes all the knowledge obtained through the design process. When a CLF is incorporated into an MPC scheme, it provides a guarantee of stability while

focusing the computations. Furthermore, the schemes that we have developed that combine CLFs and MPC are more implementable and flexible than traditional MPC approaches, since they do not require global optimum to non-convex optimizations, but rather use on-line computation to improve solutions as much as possible given the imposed time limitations. This provides a promising future paradigm for a unified approach to nonlinear optimal control.

References

[1] C. Beck. *Model Reduction for Multi-Dimensional and Uncertain Systems.* PhD thesis, California Institute of Technology, 1995.

[2] C. Beck, R. D'Andrea, F. Paganini, W. M. Lu, and J.C. Doyle. A state-space theory of uncertain systems. In *IFAC Proceedings*, 1996.

[3] J.R. Cloutier, C.N. D'Souza, and C.P.Mracek Marcek. Nonlinear regulation and nonlinear H_∞ control via the state-dependent riccati equation technique. In *Proc. 1st Internat. Conf. on Nonl. Problems in Aviation and Aeraspace*, Daytona Beach, FL, 1996.

[4] R. D'Andrea. H-infinity optimal interconnections. Technical Report 030, Caltech, 1995. http://avalon.caltech.edu/cds/, submitted to IEEE Transactions on Automatic Control.

[5] Raffaello D'Andrea and Sven Khatri. Kalman decomposition of linear fractional transformation representations and minimality. In *Proc. American Control Conference*, 1997.

[6] Raffaelo D'Andrea. *Generalizations of $\mathcal{H}_\infty$ optimization; Control of Rotating Stall.* PhD thesis, California Institute of Technology, 1997.

[7] G. Dullerud and S. Lall. Analysis and synthesis tools for time-varying systems. In *Proc. IEEE Control and Decision Conference*, 1997.

[8] G. E. Dullerud and R. S. Smith. A continuous-time extension condition. *IEEE Transactions on Automatic Control*, 1995. (to appear).

[9] G. E. Dullerud and R. S. Smith. Experimental application of time domain model validation: Algorithms and analysis. *Int. J. of Rob. and Non. Con.*, 1995. (submitted).

[10] G. E. Dullerud and R. S. Smith. The experimental validation of robust control models for a heat experiment: A linear matrix inequality approach. In *Proc. IEEE Control and Decision Conference*, 1995.

[11] R. Freeman and J. Primbs. Control lyapunov functions: New ideas from an old source. In *Proceedings of the* 35^{th} *IEEE Conference on Decision and Control*, pages 3926–3931, Kobe, Japan, December 1996.

[12] S. Glavaski and J. E. Tierno. Robustness Analysis: Nonlinear and Growing. In *Proceedings of 35th IEEE Conference on Decision and Control*, Kobe, Japan, 1997.

[13] S. Glavaški and J. E. Tierno. Advances in Worst-case $\mathcal{H}_\infty$ Performance Computation. In *Proceedings of the IEEE International Conference on Control Applications*, Trieste, Italy, 1998. (to appear).

[14] Y. Huang. *Ninlinear Optimal Control: An Enhanced Quasi-LPV Approach.* PhD thesis, California Institute of Technology, 1998.

[15] Y. Huang and W.M. Lu. Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation. In *Proc. IEEE Control and Decision Conference*, Kobe, Japan, 1996.

[16] S. Khatri, R. D'Andrea, and J. Doyle. Hierarchical uncertain modelling. In *Proc. IEEE Control and Decision Conference*, 1996.

[17] S. Khatri and P. A. Parrilo. Spherical μ. In *Proc. American Control Conference*, 1998.

[18] Sven Khatri and Pablo Parrilo. Guaranteed lower bounds for probabilistic μ. In *Proc. IEEE Control and Decision Conference*, 1998. (submitted).

[19] Sven Khatri and Pablo Parrilo. Spherical μ. In *Proc. American Control Conference*, 1998. (to appear).

[20] S. Lall, C. Beck, and G. Dullerud. Guaranteed error bounds for model reduction of linear time-varying systems. submitted to American Control Conference, 1997.

[21] S. Lall and G. Dullerud. H_∞ synthesis for multi-rate hybrid systems. In *Proc. IEEE Control and Decision Conference*, 1997.

[22] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, 1997.

[23] J. C. Morris and M. P. Newlin. Model validation in the frequency domain. In *Proc. IEEE Control and Decision Conference*, 1995.

[24] M. P. Newlin and S. T. Glavaski. Advances in the computation of the μ lower bound. In *Proc. American Control Conference*, pages 442–446, 1995.

[25] M. P. Newlin and R. S. Smith. A generalization of the structured singular value and its application to model validation. *IEEE Transactions on Automatic Control*, 1995. (to appear).

[26] F. Paganini. *Sets and Constraints in the Analysis of Uncertain Systems.* PhD thesis, California Institute of Technology, 1995.

[27] Pablo Parrilo and Sven Khatri. Closed form solutions for a class of lmis. In *Proc. American Control Conference*, 1998. (to appear).

[28] J. A. Primbs, V Nevistić, and J. C. Doyle. On Receding Horizon Extensions and Control Lyapunov Functions. In *In Proc. of the American Control Conference*, Philadelphia, Pennsylvania, 1998.

[29] R. S. Smith and G. E. Dullerud. Validation of continuous-time control models by finite experimental data. In *Proc. American Control Conference*, 1995. (submitted, IEEE T. Automatic Control).

[30] J. Tierno, R. M. Murray, and J. C. Doyle. An efficient algorithm for performance analysis of nonlinear control systems. In *Proc. American Control Conference*, 1995.

[31] J. E. Tierno and S. Glavaski. Practical Upper and Lower Bounds for Robust $\mathcal{H}_2$ Performance under LTI Perturbations. In *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Boston, MA, 1998. (to appear).

[32] J. E. Tierno and R. M. Murray. Robust performance analysis for a class of uncertain nonlinear system. In *Proc. IEEE Control and Decision Conference*, 1995.

[33] Fen Wu, Hua Xin Yang, Andy Packard, and Greg Becker. Induced L_2-Norm control for lpv system with bounded parameter variation. *Int. Journal of Nolinear and Robust Control*, vol.6:983–998, 1996.

[34] P. Young, M. Newlin, and J. Doyle. Computing bounds for the mixed μ problem. *Int'l. J. Robust and Nonlinear Control*, 5:573–590, 1995.

[35] P. Young, M. Newlin, and J. Doyle. Let's get real. In *Robust Control Theory*. Springer Verlag, 1995. pp. 143–174.

[36] P. Young, M. Newlin, J. Doyle, and A. Packard. Theoretical and computational aspects of the structured singular value. *Inst. Sys. Contr. Inf. Eng.*, 1995. (to appear).

[37] Xiaoyun Zhu, Yun Huang, and John Doyle. Soft vs. hard bounds in probabilistic robustness analysis. In *Proc. IEEE Control and Decision Conference*, pages 3412–3417, 1996.

[38] Xiaoyun Zhu, Sven Khatri, and Pablo Parrilo. μ with linear cuts: upper bound computation. In *Proc. American Control Conference*, 1999. (submitted).